Uniformization of negatively curved klt pairs

joint with
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① Smooth uniformization

X^n smooth projective ; $\Delta_{MY}(X) = 2(n+1)c_2(X) - n c_1^2(X) \in H^{2,2}(X,\mathbb{R}) \subset H^4(X,\mathbb{R})$

Thm : Assume $K_X > 0$. Then $\Delta_{MY}(X) \cdot c_1(K_X)^{n-2} \geq 0$

and equality occurs $\iff X = \mathbb{B}^n/\Gamma$, $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$ discrete acting prop. disc. w/ no fixed point.

Pf : '78. Aubin-Yau : $\exists\ \omega$ Kähler metric : $\mathrm{Ric}(\omega) = -\omega$.
$(\omega \in c_1(K_X))$

'75. Chen-Ogiue

$$\int_X \Delta_{MY}(X,\omega) \wedge \omega^{n-2} = \int_X \Big(a_n \underbrace{|\mathring{\Theta}(T_X,\omega)|^2_\omega}_{\geq 0} - b_n \cancel{|\mathring{\mathrm{Ric}}(\omega)|^2_\omega}\Big) \cdot \omega^n$$

$\Theta(T_X,\omega)$ Chern curvature of ω : $\in C^\infty(\Lambda^{1,1}T_X \otimes \mathrm{End}\, T_X)$

$\mathring{\Theta}(T_X,\omega) = \Theta(T_X,\omega) + \frac{1}{n}\, \omega \otimes \mathrm{Id}_{T_X}$

$\mathring{\mathrm{Ric}}(\omega) = \mathrm{Ric}\,\omega + \omega = 0$

≥ 0 : ok

$= 0$: implies $\Theta(T_X,\omega) = -\frac{1}{n} \cdot \omega \otimes \mathrm{Id}_{T_X}$: constant curvature

$\Rightarrow (\widetilde{X}, \pi^*\omega) \simeq (\mathbb{B}^n, \omega_{\mathrm{Berg}})$

② Orbifold uniformization

$X = \mathbb{B}^n/\Gamma$ with Γ discrete, acting prop. disc. but maybe not freely.
Assume X compact.

Two things go wrong:
- X may be singular
- K_X may not be ample

Selberg lemma. $\exists\, \Gamma' \leq \Gamma$ finite index s.t. $\Gamma' \circlearrowright \mathbb{B}^n$ has no fixed point.

$$\begin{array}{ccccc} \mathbb{B}^n & \xrightarrow[\text{étale}]{} & \mathbb{B}^n/\Gamma' & \xrightarrow[\text{finite}]{f} & \mathbb{B}^n/\Gamma \\ & & \| & & \| \\ & & Y & \longrightarrow & X \end{array}$$

f is ramified in gal. Ramification in codimension 1, $\exists$ divisors $\Delta_1, \dots, \Delta_N$ s.t. f ramifies at order $m_i \geq 2$ along Δ_i

$\Delta_\Gamma := \sum (1 - \frac{1}{m_i}) \Delta_i$ $\mathbb{Q}$-divisor on X

By def $\boxed{K_Y = f^*(K_X + \Delta)}$

def. In case above, one says that $(\mathbb{B}^n/\Gamma, \Delta_\Gamma)$ is a ball quotient.

def. Complex orbifold = quasi-projective pair (X, Δ) where:

① X normal q proj variety, $\Delta = \sum (1 - \frac{1}{m_i}) \Delta_i$ $\mathbb{Q}$-divisor, $m_i \in \mathbb{N}_{\geq 0}$.

② $\exists$ covering (X_α) of X w/ finite Galois covers $f_\alpha : Y_\alpha \to X_\alpha$
- $\to$ Y_α is smooth
- $\to$ $K_{Y_\alpha} = f_\alpha^*(K_{X_\alpha} + \Delta|_{X_\alpha})$

def. • Orbifold diff form γ on (X, Δ) = smooth diff form on $X_{reg} \setminus |\Delta|$ s.t. $f_\alpha^* \gamma$ extends to a smooth form on Y_α

- Similar notion for orbifold Kähler form
- Important obs: if γ is a $2n$-form, then $\int_{X_{reg}\setminus|\Delta|} \gamma$ is convergent.

<u>Consequences</u>:
- can define de Rham coh. groups $H^k_{dR}(X,\mathbb{R}) \simeq H^k_{sing}(X,\mathbb{R})$
- Chern-Weil formalism $c_i(X,\Delta) \in H^{2i}_{dR}(X,\mathbb{R})$

  represented by $c_i(X_{reg}\setminus|\Delta|, \omega)$ for orb. Kähler form

<u>Thm</u> (X,Δ) proj orbifold s.t. $K_X+\Delta$ ample.
Then $\Delta_{MY}(X,\Delta)\cdot c_1(K_X+\Delta)^{n-2} \geq 0$, eq. $\Leftrightarrow$ (X,Δ) is a ball quotient.

<u>Proof</u>: Same as above w/ one difference

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③ <u>klt uniformization</u>

<u>def</u>. A klt pair (X,Δ) is a pair

① X = q-proj variety, Δ = effective $\mathbb{Q}$-div (coeff in $[0,1)$).

② $K_X+\Delta$ is $\mathbb{Q}$-Cartier & if Ω is a local trivialization of $m(K_X+\Delta)$

$$\int_{\substack{X_{reg}\setminus|\Delta| \\ (loc)}} i^{n^2} (\Omega\wedge\bar{\Omega})^{1/m} < +\infty$$

(Complex orbifolds (X,Δ) are klt)

<u>Prop</u>: If Δ has standard $(1-\frac{1}{m})$ coeff, then $\exists\, Z \subsetneq X$ $\operatorname{codim}_X Z \geq 3$ s.t. if $X^\circ = X\setminus Z$, $\Delta^\circ = \Delta|_{X\setminus Z}$, then (X°,Δ°) is a complex orbifold

Proof. Reduce to $\Delta = 0$ using local cyclic coverings.

$\hookrightarrow$ treated explicitely by Greb - Kebekus - Kovacs - Peternell

comes down to klt surface is an orbifold (classification)

Consequence : If (X, Δ) is a projective klt pair, one can define

- $c_2(X, \Delta) \cdot c_1(L)^{n-2}$ $\forall$ $\mathbb{Q}$-line bundle L
  $\underset{\mathbb{Q}}{\cap}$
- $c_1^2(X, \Delta) \cdot c_1(L)^{n-2}$

Thm (Claudon - G - Graf '23). (X, Δ) proj klt pair w/ st. coeff. s.t. $K_X + \Delta$ ample

Then $\Delta_{MY}(X, \Delta) \cdot c_1(K_X + \Delta)^{n-2} \geq 0$, equality occurs iff (X, Δ) is a ball quotient.

Rk : ① Ineq. not new (G - Taji '16 in dlt case $K_X + \Delta$ nef)

② Case $\Delta = 0$ obtained by GKP - Taji '15, 19, '20 .

Ingredients : ① T_X is semistable wrt K_X (uses singular KE metrics)

② topological . if (X, Δ) klt, $|\pi_1(X_{reg})| < +\infty$

Xu - GKP '16 - Braun '19
'14

③ Simpson's approach using Higgs bundles

$$\Delta_{MY}(X) = c_2(\mathrm{End}(T_X \oplus \mathcal{O}_X))$$

④ Zariski - Lipman conj proved in klt case (Druel - GKKP)

X smooth $\Leftrightarrow$ T_X is locally free

Proof. Ideally we'd like to find $Y \xrightarrow{f} X$ s.t.

$$K_Y = f^0(K_X + \Delta)$$

$\Rightarrow Y$ klt, and $\Delta_{BY}(Y) \cdot K_Y^{n-2} = 0$

Crux. Cyclic coverings; we can find f s.t.

- f ramifies at order m_i along Δ_i
- Extra ramification in codim 1 is supported along a general element of a very ample linear system $|H|$.

$$K_Y = f^0\left(K_X + \Delta + \left(1 - \frac{1}{N}\right) H\right)$$

$\Rightarrow Y$ klt + H can be moved.

Show that X has quotient sing $\Leftarrow Y \setminus f^{-1}(H)$ has quot. sing.

Prove: $f^{\#} T_{(X,\Delta)}$ is semistable wrt. $f^0(K_X + \Delta)$

T_Y (w/ equality away from $f^{-1}(H)$)

$\boxed{\text{Using GKPT}}$: $f^{\#} T_{(X,\Delta)}$ becomes loc. free on a finite q. étale cover of $Y \Rightarrow Y \setminus f^{-1}(H)$ has quotient sing.

$X = \mathbb{B}^4 / \Gamma$ $\quad \Gamma$ acts freely with finite covolume

$X \hookrightarrow \overline{X}$ toroidal compactification.

$$\overline{X} = X \sqcup \overbrace{\bigsqcup_{i=1} D_i}^{D} \quad \sim \text{ab. var.}$$

$$2(n+1)\, c_2(\bar{X}, D) - n\, c_1^2(\bar{X}, D) \equiv 0.$$

($K_{\bar{X}} + D$ semiample)